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GRAPH THEORETIC METHOD FOR DETERMINING HURWITZ EQUIVALENCE IN THE SYMMETRIC GROUP

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ABSTRACT

Motivated by the problem of Hurwitz equivalence of Δ^2 factorization in the braid group, we address the problem of Hurwitz equivalence in the symmetric group, of 1_{S_n} factorizations with transposition factors. Looking at the transpositions as the edges in a graph, we show that two factorizations are Hurwitz equivalent if and only if their graphs have the same weighted connected components. The graph structure allows us to compute Hurwitz equivalence in the symmetric group. Using this result, one can compute non-Hurwitz equivalence in the braid group.

1. Definitions

Definition 1.1. Hurwitz move on G^m (R_k, R_k^{-1}): Let G be a group, $\overrightarrow{t} = (t_1, \ldots, t_m) \in G^m$. We say that $\overrightarrow{s} = (s_1, \ldots, s_m) \in G^m$ is obtained from $\overrightarrow{t}$ by the Hurwitz move R_k (or $\overrightarrow{t}$ is obtained from $\overrightarrow{s}$ by the Hurwitz move R_k^{-1}) if

$$s_i = t_i \quad \text{for } i \neq k, k+1,$$
$$s_k = t_k t_{k+1} t_k^{-1}, \quad s_{k+1} = t_k.$$

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Notation 1.2. Factorization: Let G be a group. A factorization F of $g \in G$ is a list of factors $f_1, f_2, \ldots, f_n$ ($f_i \in G$) such that the product $f_1 f_2 \ldots f_n$ is equal to g.

We use the notation $f_1 \cdot f_2 \cdots f_n$ for the factorization $f_1, f_2, \ldots, f_n$ and $f_1 f_2 \ldots f_n$ for the product.

Definition 1.3. Hurwitz move on factorization: Let G be a group and $t \in G$. Let $t = t_1 \cdots t_m = s_1 \cdots s_m$ be two factorized expressions of t. We say that $s_1 \cdots s_m$ is obtained from $t_1 \cdots t_m$ by the Hurwitz move R_k if $(s_1, \ldots, s_m)$ is obtained from $(t_1, \ldots, t_m)$ by the Hurwitz move R_k.

Definition 1.4. Hurwitz equivalence of factorization: The factorizations $s_1 \cdots s_m$, $t_1 \cdots t_m$ are Hurwitz equivalent if they are obtained from each other by a finite sequence of Hurwitz moves. The notation is $t_1 \cdots t_m \overset{HE}{\backsim} s_1 \cdots s_m$.

2. Hurwitz equivalence properties in S_n

Definition 2.1: Let $F = \Gamma_1 \cdots \Gamma_m$ be a factorization such that, $\Gamma_i = (a_i, b_i)$ and $1 \leq a_i, b_i \leq n$. We define $G_F = (V_F, E_F)$ to be the graph of the factorization F, where $V_F = \{1, \ldots, n\}$ are the vertices and $E_F = \{(i,j) | \exists k \text{s.t. } \Gamma_k = (i,j)\}$ are the edges of the factorization graph.

Definition 2.2: We define the weight of an edge $(i,j) \in E_F$ as the number of elements Γ_k s.t. $\Gamma_k = (i,j)$. The weight of (i,j) in the factorization F will be noted as $W_F((i,j))$.

For a given graph G_F we denote the graphs of its connected components as $G_F^1, \ldots, G_F^d$, where d is the number of connected components in the graph. For each connected component, let $G_F^i = (V_F^i, E_F^i)$, where V_F^i are the vertices of G_F^i and E_F^i are the edges.

Definition 2.3: We define the weight of the connected component G_F^i to be

$$W(G_F^i) = \sum_{e_r \in E_F^i} W(e_r).$$

THEOREM 2.4: *Let X be the set of all m-tuples $(t_1, \ldots, t_m)$, such that:*

1. *t_i is a transposition in S_n,*
2. *$t_1 \ldots t_m = 1$,*
3. *the group generated by $t_1, \ldots, t_m$ operates transitively on $\{1, \ldots, n\}$.*

Then, all the elements in X are Hurwitz equivalent.

The following corollary is equivalent, using graph definitions:

COROLLARY 2.5: *All factorizations, $t_1 \cdots t_m$, such that:*

1. *t_i is a transposition in S_n,*
2. *$t_1 \dots t_m = 1$,*
3. *The factorization has a single connected component equal to $\{1, \dots, n\}$,*

are Hurwitz equivalent.

Corollary 2.5 is equivalent to Theorem 2.4, since the condition, "single connected component", and the condition, "The group generated by $t_1, \dots, t_m$ operates transitively", are equivalent.

The following is a generalization of Theorem 2.4 for non-transitive cases:

THEOREM 2.6: *Let F_1, F_2 be two 1_{S_n} factorizations with the same number of factors. Then $F_1 \overset{HE}{\backsim} F_2$ if and only if G_{F_1} and G_{F_2} have the same number of connected components $G^1_{F_1}, \dots, G^d_{F_1}$ and $G^1_{F_2}, \dots, G^d_{F_2}$ respectively, and there exists a permutation π s.t. $V^i_{F_1} = V^{\pi(i)}_{F_2}$ and $W(G^i_{F_1}) = W(G^{\pi(i)}_{F_2})$ for each $i \leq d$.*

In other words, two factorizations are Hurwitz equivalent if and only if the connected components of their factorization graphs contain the same nodes and have the same weights.

Example 2.7: The 1_{S_n} factorizations,

$$F_1 = (2,6) \cdot (1,4) \cdot (1,5) \cdot (3,6) \cdot (4,5) \cdot (1,5) \cdot (2,3) \cdot (3,6),$$
$$F_2 = (2,6) \cdot (1,5) \cdot (3,6) \cdot (3,6) \cdot (2,6) \cdot (1,5) \cdot (1,4) \cdot (1,4),$$

have connected components with the same nodes and and weights, as shown in Figure 1, and by Theorem 2.6 they are Hurwitz equivalent.

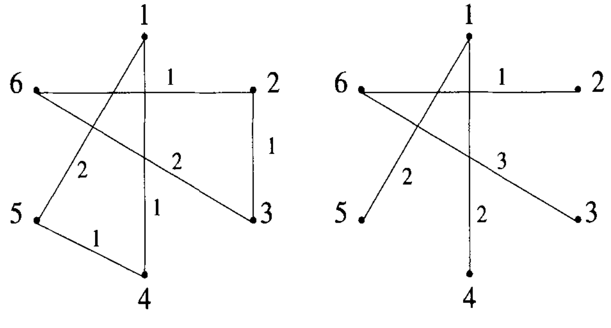

Figure 1. The graphs of the factorizations F_1 and F_2.

The rest of the section will be devoted to the proof of Theorem 2.6, starting with the first direction of the theorem.

Proof of the first direction: In the proof of the first direction, we prove that if two factorizations are Hurwitz equivalent, the factorizations have the same graph components with the same weights. Therefore, it is sufficient to show that when operating a single Hurwitz move, the vertices and weights of the graph's connected components will remain the same.

Let $F_1 = \Gamma_1 \cdots \Gamma_m$ and let $F_2 = \Gamma_1 \cdots \Gamma_i \Gamma_{i+1} \Gamma_i^{-1} \cdot \Gamma_i \cdots \Gamma_m$ be the factorization obtained from F_1 by performing Hurwitz move R_i.

Let $\Gamma_j = (a_j, b_j)$, $1 \leq a_j, b_j \leq n$, $j \leq m$; then, in the cases where

$$\{a_i, b_i\} \bigcap \{a_{i+1}, b_{i+1}\} = \emptyset \quad \text{or} \quad \{a_i, b_i\} \bigcap \{a_{i+1}, b_{i+1}\} = \{a_i, b_i\},$$

we get that

$$\Gamma_i \Gamma_{i+1} \Gamma_i^{-1} = \Gamma_{i+1}$$

and the two factorizations have the same factors in a different order, so the factorization graphs G_{F_1} and G_{F_2} are the same.

In the case where $\Gamma_i = (a_i, b)$ and $\Gamma_{i+1} = (a_{i+1}, b)$, $\Gamma_i \Gamma_{i+1} \Gamma_i^{-1} = (a_i, a_{i+1})$ replaces Γ_{i+1}, so the nodes of the connected components remain the same and so do their weights.

Proof of the second direction: To complete Theorem 2.6, we need to show that if two factorizations have the same connected components with the same weights they are Hurwitz equivalent. To prove that, we will show that each factorization is Hurwitz equivalent to a standard canonical factorization which depends only on the nodes of the factorization's connected components and their weights.

LEMMA 2.8: *Let $(a, b) \cdot (c, d)$ be a factorization in S_n. Then:*

1. *By performing Hurwitz move R_0 we get*

$$(a,b)\cdot(c,d) \overset{HE}{\backsim} \begin{cases} (c,d)\cdot(a,b), & \text{if } \{a,b\}\bigcap\{c,d\} = \emptyset, \\ (c,d)\cdot(a,b), & \text{if } \{a,b\}\bigcap\{c,d\} = \{a,b\}, \\ (a,d)\cdot(a,b), & \text{if } b = c \text{ and } a \neq d. \end{cases}$$

2. *By performing Hurwitz move ${R_0}^{-1}$ we get*

$$(a,b)\cdot(c,d) \overset{HE}{\backsim} \begin{cases} (c,d)\cdot(a,b), & \text{if } \{a,b\}\bigcap\{c,d\} = \emptyset, \\ (c,d)\cdot(a,b), & \text{if } \{a,b\}\bigcap\{c,d\} = \{a,b\}, \\ (c,d)\cdot(a,d), & \text{if } b = c \text{ and } a \neq d. \end{cases}$$

3. $(a,b)\cdot 1_{S_n} \overset{HE}{\backsim} 1_{S_n} \cdot (a,b)$.

Proof: Trivial.

Let $F_1 = \Gamma_1 \cdots \Gamma_m$ be a factorization of 1_{S_n}, where all factors are of transpositions.

LEMMA 2.9: *If* $\Gamma_j \in E_{F_1}^{t_1}$ *and* $\Gamma_{j+1} \in E_{F_1}^{t_2}$, $t_1 \neq t_2$ *then*

$$\Gamma_1 \cdots \Gamma_j \cdot \Gamma_{j+1} \cdots \Gamma_m \overset{HE}{\backsim} \Gamma_1 \cdots \Gamma_{j+1} \cdot \Gamma_j \cdots \Gamma_m.$$

Proof: Since Γ_j, Γ_{j+1} belong to a different connected component, they do not connect the same vertex and therefore, $\Gamma_j \Gamma_{j+1} \Gamma_j^{-1} = \Gamma_{j+1}$ (see Lemma 2.8). Therefore, by operating Hurwitz move R_i they commute.

As a result, for each connected component, all elements of the component commute with all elements of other components. Therefore, factorization is Hurwitz equivalent to a factorization with the same factors ordered according to the component to which they belong. For example, order the connected components by the lowest vertex they contain, then gather all factors of the first component to the left, and after them the factors of the second component and so on.

Let $\{G_{F_1}^r\}_{r=1}^s$ be the distinct connected components of G_{F_1}. From Lemma 2.9, $F_1 \overset{HE}{\backsim} f_1 \cdots f_s$, where f_r is a factorization with elements from $E_{F_1}^r$. The length of the factorization f_r is equal to $W(G_{F_1}^r)$ and s is the number of connected components. Therefore, to conclude the proof, it is sufficient to show that each f_r is Hurwitz equivalent to a standard canonical factorization which depends only on the length of f_r (which can never be changed by Hurwitz moves) and $V_{F_1}^r$.

We define an order on $V_{F_1}^r$ vertices, $V_{F_1}^r = \{v_{t_1}, \ldots, v_{t_l}\}$. Note that since $F_1 = 1_{S_n}$, $f_r = 1_{S_n}$ as a product.

LEMMA 2.10: *Let* $f = \Gamma_1 \cdots \Gamma_m$ *be a factorization with a single connected component,* G_f^1*; then* $\forall v_1, v_2 \in V_f^1$, $f \overset{HE}{\backsim} (v_1, v_2) \cdot \gamma_1 \cdots \gamma_{m-1}$.

Proof: By induction on the minimal length of the path connecting v_1 with v_2. In the case where the minimal length is 1, there exists $1 \leq j \leq m$ s.t. $\Gamma_j = (v_1, v_2)$. Operating $\{R_k{}^{-1}\}_{k=j-2}^0$ sequence of Hurwitz moves, we get a factorization $(v_1, v_2) \cdot \gamma_1 \cdots \gamma_{m-1}$ (see Lemma 2.8). We will assume that the lemma is true for a path with length less than n, and we will prove that the factorization where the minimal path between v_1 and v_2 is n, is Hurwitz equivalent to a factorization for which the path between v_1 and v_2 is of length $n-1$.

Let $(a_1, a_2), (a_2, a_3), \ldots, (a_n, a_{n+1})$ be the minimal path between $v_1 = a_1$ and $v_2 = a_{n+1}$. To prove the above we will perform another induction, on the number of factors which are in between (a_1, a_2) and (a_2, a_3). We will assume that (a_1, a_2) is left to (a_2, a_3):

$$f = \cdots (a_1, a_2) \cdot (b_1, c_1) \cdot (b_2, c_2) \cdots (b_k, c_k) \cdot (a_2, a_3) \cdots.$$

Let k be the number of factors between (a_1, a_2) and (a_2, a_3). In the case where $k = 0$, $(a_1, a_2) \cdot (a_2, a_3) \overset{HE}{\backsim} (a_1, a_3), (a_1, a_2)$ and we are done, since the new factorization contains the path $(a_1, a_3), (a_3, a_4), \ldots, (a_n, a_{n+1})$, which is of length $n - 1$.

In the case where $k > 0$:

- If $\{a_1, a_2\} \bigcap \{b_1, c_1\} = \emptyset$, then $(a_1, a_2) \cdot (b_1, c_1) \overset{HE}{\backsim} (b_1, c_1) \cdot (a_1, a_2)$ (by Lemma 2.8), and now (a_1, a_2) and (a_2, a_3) are separated by $k - 1$ factors, and so we are done.
- If $a_1 = b_1$, then $(a_1, a_2) \cdot (a_1, c_1) \overset{HE}{\backsim} (a_2, c_1) \cdot (a_1, a_2)$ (by Lemma 2.8), and now (a_1, a_2) and (a_2, a_3) are separated by $k - 1$ factors. Note that (a_1, c_1) is not an element in the path (since the path is minimal).
- If $a_2 = b_1$, then $(a_1, a_2) \cdot (a_2, c_1) \overset{HE}{\backsim} (a_1, c_1) \cdot (a_1, a_2)$ (by Lemma 2.8), and now (a_1, a_2) and (a_2, a_3) are separated by $k - 1$ factors. Note that if (a_2, c_1) is in the path, then $c_1 = a_3$ and then $k = 0$.

This concludes the proof of Lemma 2.10.

LEMMA 2.11:

1. $(a, b) \cdot (a, b) \cdot (a, c) \cdot (a, c) \overset{HE}{\backsim} (a, c) \cdot (a, c) \cdot (a, b) \cdot (a, b)$.
2. $(a, b) \cdot (a, b) \cdot (a, c) \cdot (a, c) \overset{HE}{\backsim} (a, b) \cdot (a, b) \cdot (b, c) \cdot (b, c)$.

Proof of 1: $(a, b) \cdot (a, b) \cdot (a, c) \overset{HE}{\backsim} (a, c) \cdot (a, b) \cdot (a, b)$. By operating Hurwitz moves R_1 and R_0, and therefore $(a, b) \cdot (a, b) \cdot (a, c) \cdot (a, c) \overset{HE}{\backsim} (a, c) \cdot (a, c) \cdot (a, b) \cdot (a, b)$.

Proof of 2: By performing the Hurwitz moves $R_1^{-1}, R_2^{-1}, R_1^{-1}$.

Now we are ready to start forming f_r into a standard canonical form: $v_{t_1}, v_{t_2} \in V_f^r$, from Lemma 2.10,

$$f_r \overset{HE}{\backsim} (v_{t_1}, v_{t_2}) \cdot f_r^1,$$

where f_r^1 is the factorization with the $W(G_{f_r}^1) - 1$ other factors.

$f_r^1 = (v_{t_1}, v_{t_2})$ as a product, since $f_r = 1_{S_n}$ and $f_r^1 = (v_{t_1}, v_{t_2})^{-1} f_r$.

Because $f_r^1 = (v_{t_1}, v_{t_2})$, f_r^1 contains a path connecting v_{t_1} with v_{t_2}. Again, using Lemma 2.10 we get

$$f_r \overset{HE}{\backsim} (v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdot f_r^2 \quad \text{and} \quad f_r^2 = 1_{S_n}.$$

By the first direction of Theorem 2.6, $(v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdot f_r^2$ still creates a single connected component. Therefore, there is a path between v_{t_2} and v_{t_3}, which means that f_r^2 contains a path from v_{t_2} to v_{t_3} or from v_{t_1} to v_{t_3} (for example, in some cases where the path in G_f^r includes (v_{t_1}, v_{t_2})).

In the first case we get

$$f_r \overset{HE}{\backsim} (v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdot (v_{t_2}, v_{t_3}) \cdot (v_{t_2}, v_{t_3}) \cdot f_r^4$$

and in the second case we get

$$f_r \overset{HE}{\backsim} (v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_3}) \cdot (v_{t_1}, v_{t_3}) \cdot f_r^4$$

which by Lemma 2.11 is Hurwitz equivalent to the first case.

We continue with this process to bring f_r to the form

$$(v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdot (v_{t_2}, v_{t_3}) \cdot (v_{t_2}, v_{t_3}) \cdots (v_{t_{m-1}}, v_{t_m}) \cdot (v_{t_{m-1}}, v_{t_m}) \cdot f_r^{2m-2}.$$

Assume we have come to the point where

$$f_r \overset{HE}{\backsim} (v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdots (v_{t_{k-1}}, v_{t_k}) \cdot (v_{t_{k-1}}, v_{t_k}) \cdot f_r^{2k-2}$$

where $k < m$ and f_r^{2k-2} is a factorization with $W(G_f^r) - 2k + 2$ factors.

Just as before, the new factorization creates a connected graph and $f_r^{2k-2} = 1_{S_n}$. Since the graph is connected, f_r^{2k-2} contains a path from $v_{t_{k+1}}$ to one of the vertices v_{t_s} $s \leq k$. Thus, there is a path from v_{t_s} $(s \leq k)$ to $v_{t_{k+1}}$ which does not include the factors left to f_r^{2k-2} because they create a connected graph which does not include $v_{t_{k+1}}$. So, from Lemma 2.10,

$$f_r \overset{HE}{\backsim} (v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdots (v_{t_{k-1}}, v_{t_k}) \cdot (v_{t_{k-1}}, v_{t_k}) \cdot (v_{t_{k+1}}, v_{t_s}) \cdot f_r^{2k-1},$$

and since $f_r^{2k-1} = (v_{t_{k+1}}, v_{t_s})$ as a product, there is a path from $v_{t_{k+1}}$ to v_{t_s}.

By Lemma 2.10,

$$f_r \overset{HE}{\backsim} (v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdots (v_{t_{k-1}}, v_{t_k}) \cdot (v_{t_{k-1}}, v_{t_k}) \cdot (v_{t_{k+1}}, v_{t_s}) \cdot (v_{t_{k+1}}, v_{t_s}) \cdot f_r^{2k}.$$

Now, only using the factors left to f_r^{2k} we need to change $(v_{t_{k+1}}, v_{t_s}) \cdot (v_{t_{k+1}}, v_{t_s})$ to $(v_{t_k}, v_{t_{k+1}}) \cdot (v_{t_k}, v_{t_{k+1}})$. Since $s \leq k$, there is a path from v_{t_s} to v_{t_k} in the graph created by the factors on the left. From this fact and using Lemma 2.11 we see that the factorizations

$$(v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdots (v_{t_{k-1}}, v_{t_k}) \cdot (v_{t_{k-1}}, v_{t_k}) \cdot (v_{t_s}, v_{t_{k+1}}) \cdot (v_{t_s}, v_{t_{k+1}}) \quad \text{and}$$
$$(v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdots (v_{t_{k-1}}, v_{t_k}) \cdot (v_{t_{k-1}}, v_{t_k}) \cdot (v_{t_k}, v_{t_{k+1}}) \cdot (v_{t_k}, v_{t_{k+1}})$$

are Hurwitz equivalent, since Lemma 2.10 allows us to commute couples of transpositions, or to change one vertex in the couple if the two couples have a common vertex.

To complete the proof of Theorem 2.6 we need to show that we can also bring the right factors, f_r^{2m-2}, to a standard form. This can be done in a similar way to the above procedure:

Take the first factor in f_r^{2m-2}, i.e.

$$f_r^{2m-2} = (v_{t_x}, v_{t_y}) \cdot f_r^{2m-1},$$

and again by using Lemma 2.10 we get

$$f_r^{2m-2} \overset{HE}{\backsim} (v_{t_x}, v_{t_y}) \cdot (v_{t_x}, v_{t_y}) \cdot f_r^{2m}.$$

Using Lemma 2.11 and the factors on the left, we can change the factors $(v_{t_x}, v_{t_y}) \cdot (v_{t_x}, v_{t_y})$ to $(v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2})$, since the graph of the factors on the left is the same as f_r. This concludes the proof of Theorem 2.6 since every factorization f_r is Hurwitz equivalent to

$$(v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdots (v_{t_{m-1}}, v_{t_m}) \cdot (v_{t_{m-1}}, v_{t_m}) \cdot (v_{t_1}, v_{t_2}) \cdots (v_{t_1}, v_{t_2}),$$

which depends only on the factorization graph and the number of factors in the factorization. ∎

3. Applications

Although the number of Hurwitz equivalence factorizations is finite, by trying to solve the problem directly we get an exponential solution. Theorem 2.6 reduces the Hurwitz equivalence problem to the connected component problem, which is linear.

Theorem 2.6 allows us to determine non-Hurwitz equivalence in bigger groups with homomorphism to the symmetric group, in particular in the braid group.

This is significant, since Hurwitz equivalence in the braid group is required to compute the BMT invariant, presented in [1] and [2]. The graph structure gives us a weaker, but easy to compute, invariant to distinguish among diffeomorphic surfaces which are not a deformation of each other.

The BMT invariant is a Δ^2 (the generator of the braid group center) factorization, with half-twist (conjugation of the braid group generators) powers as factors.

When projecting the BMT factorization to the symmetric group, we get a 1_{S_n} factorization with transposition factors and 1_{S_n} factors (for the even powers). By Lemma 2.8, the 1_{S_n} factors commute with all other factors and therefore, when computing Hurwitz equivalence for factorizations with 1_{S_n} factors, we require that the two factorizations have the sme number of 1_{S_n} factors and compute Hurwitz equivalence for the transposition factors, ignoring the 1_{S_n} factors.

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